Gaussian Elimination

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With the help of matrix multiplication, a system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ .... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{cases} \tag{0.1}$$

can now be expressed as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ ... \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ ... \\ b_m \end{bmatrix}$$

Now we are going to introduce a rule that can solve the system of equations by playing with matrices.

Theorem 1. *Given a system of linear equations as above, we can solve it by the following steps:*

1. *Write down a matrix*

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & b_m \end{array}\right]$$

2. *permute the rows of the matrix in the previous step so that* $a_{11} \neq 0$. *(If the first column is originally all zeros, proceed to dealing with the second column directly)*

3. *Divide the first row by* a_{11}

4. *Subtract from the i-th row* ($i \geq 2$) *by* a_{i1} *times the first row. Then we will obtain a matrix with the first column to be zeros except the first row.*

5. *If* $a_{22} = 0$, *switch the second row with a* ***lower*** *row to make* $a_{22} =\neq 0$. *(If we cannot find a nonzero one, then proceed to dealing with the third column directly.)*

6. *Divide the second row by* a_{22}.

7. *Subtract the i-th row except the second by* a_{i2} *times the secondrow.*

8. *Then repeat the above process for the* j*-th column for each* j *in a analogue way. We finally obtain a matrix that implies solutions clearly.*

Example 2. *Solve the system of equations*

$$\begin{cases} 2x_2 - x_3 = -7 \\ x_1 + x_2 + 3x_3 = 2 \\ -3x_1 + 2x_2 + 2x_3 = -10 \end{cases}$$

We first form the matrix

$$\left[\begin{array}{ccc|c} 0 & 2 & -1 & -7 \\ 1 & 1 & 3 & 2 \\ -3 & 2 & 2 & -10 \end{array}\right]$$

We start dealing with the first column. Note that $a_{11} = 0$, *and* $a_{21} = 1 \neq 0$, *we switch the first two rows to make* $a_{11} \neq 0$*:*

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 2 & -1 & -7 \\ -3 & 2 & 2 & -10 \end{array}\right]$$

Now since $a_{11} = 1$ *and* a_{21} *is already* 0, *we make* a_{31} *zero by subtracting* -3 *times the first row to the third row:*

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 2 & -1 & -7 \\ -3-(-3)\times 1 & 2-(-3)\times 1 & 2-(-3)\times 3 & -10-(-3)\times 2 \end{array}\right]$$

i,e.,

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 2 & -1 & -7 \\ 0 & 5 & 11 & -4 \end{array}\right]$$

Next, we deal with the second column. Since $a_{22} = 2 \neq 0$, *we divide the second row by* 2 *to make* $a_{22} = 1$*:*

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 5 & 11 & -4 \end{array}\right]$$

Then we subtract the third row by $a_{32} = 5$ *times of the second row:*

$$\left[\begin{array}{ccc|c} 1 & 1-1\times 1 & 3-1\times(-\frac{1}{2}) & 2-1\times(-\frac{7}{2}) \\ 0 & 1 & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 5-5\times 1 & 11-5\times(-\frac{1}{2}) & -4-5\times(-\frac{7}{2}) \end{array}\right]$$

i.e.,

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{2} & \frac{11}{2} \\ 0 & 1 & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 0 & \frac{27}{2} & \frac{27}{2} \end{array}\right]$$

At last we deal with the last column by first making $a_{33} = 1$*:*

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{2} & \frac{11}{2} \\ 0 & 1 & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Then subtract the first and second row by a_{i3} *times the third row:*

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{2}-(\frac{7}{2}\times 1) & \frac{11}{2}-(\frac{7}{2}\times 1) \\ 0 & 1 & -\frac{1}{2}-(-\frac{1}{2}\times 1) & -\frac{7}{2}-(-\frac{1}{2}\times 1) \\ 0 & 0 & 1 & 1 \end{array}\right]$$

i.e.,

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

The above matrix means the original system of equations has been reduced to the following:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

i.e.,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

So the solution is $x_1 = 2, x_2 = -3, x_3 = 1$

Remark 3. The three types of moves appear in this algorithm (switching rows, multiplying or dividing a row by a number, and add or subtract to a row a multiple of another row) are called **elementary row operations**.

If two matrices A and B are related by an elementary row operation, we can write $A \sim B$. So the computation in the above example can be expressed as:

$$\left[\begin{array}{ccc|c} 0 & 2 & -1 & -7 \\ 1 & 1 & 3 & 2 \\ -3 & 2 & 2 & -10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 2 & -1 & -7 \\ -3 & 2 & 2 & -10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 2 & -1 & -7 \\ 0 & 5 & 11 & -4 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 5 & 11 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{2} & \frac{11}{2} \\ 0 & 1 & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 0 & \frac{27}{2} & \frac{27}{2} \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{2} & \frac{11}{2} \\ 0 & 1 & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 0 & 1 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

The goal of Gaussian Elimination is to repeatedly apply the three elementary moves to the matrix representing the equations, to reduce it to the reduced row echelon form defined below.

Definition 4. A matrix is in echelon form if:

1. The row of all zeros (if any) are below the other rows.
2. The first nonzero entry in each row is 1.
3. The first 1 in the $(i+1)$-th row is to the right of the first 1 in the i-th row for each i.
4. All the other entries in the column of a leading 1 in each row are zero.

Sometimes the system of equations may have more than one solution, or have no solution.

Example 5. *Solve the following system of equations*

$$\begin{cases} x_1 - 2x_2 + x_3 + 2x_4 = a \\ x_1 + x_2 - x_3 + x_4 = b \\ x_1 + 7x_2 - 5x_3 - x_4 = c \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & a \\ 1 & 1 & -1 & 1 & b \\ 1 & 7 & -5 & -1 & c \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & a \\ 0 & 3 & -2 & -1 & b-a \\ 0 & 9 & -6 & -3 & c-a \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & a \\ 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & \frac{b-a}{3} \\ 0 & 9 & -6 & -3 & c-a \end{array}\right]$$
$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -\frac{1}{3} & \frac{4}{3} & \frac{1}{3}(a+2b) \\ 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & \frac{b-a}{3} \\ 0 & 0 & 0 & 0 & 2a-3b+c \end{array}\right]$$

Since the last row of the last matrix is all 0*, the system of equations have solutions only if* $2a - 3b + c = 0$*, i.e.* $c = 3b - 2a$*.*

The first two rows implies

$$\begin{cases} x_1 - \frac{1}{3}x_3 + \frac{4}{3}x_4 = \frac{1}{3}(a+2b) \\ x_2 - \frac{2}{3}x_3 - \frac{1}{3}x_4 = \frac{b-a}{3} \end{cases}$$

We see each choice of x_3 *and* x_4 *determine* x_1 *and* x_2*, so let* $x_3 = s, x_4 = t$*, we obtain all solutions to be*

$$\begin{cases} x_1 = \frac{1}{3}(a+2b) + \frac{1}{3}s - \frac{4}{3}t \\ x_2 = \frac{1}{3}(b-a) + \frac{2}{3}s + \frac{1}{3}t \\ x_3 = s \\ x_4 = t \end{cases}$$

where s, t *can be any real numbers.*

Example 6. *Solve the system of linear equations*

$$\begin{cases} x_1 + 2x_2 - 3x_3 + 5x_4 + 2x_5 = 8 \\ 2x_1 + 4x_2 - 8x_3 + 6x_4 - 6x_5 = 0 \\ x_3 + 2x_4 + 4x_5 = -2 \end{cases}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & -3 & 5 & 2 & 8 \\ 2 & 4 & -8 & 6 & -6 & 0 \\ 0 & 0 & 1 & 2 & 4 & -2 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 2 & -3 & 5 & 2 & 8 \\ 0 & 0 & -2 & -4 & -10 & -16 \\ 0 & 0 & 1 & 2 & 4 & -2 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 2 & -3 & 5 & 2 & 8 \\ 0 & 0 & 1 & 2 & 5 & 8 \\ 0 & 0 & 1 & 2 & 4 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 11 & 17 & 32 \\ 0 & 0 & 1 & 2 & 5 & 8 \\ 0 & 0 & 0 & 0 & -1 & -10 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 11 & 17 & 32 \\ 0 & 0 & 1 & 2 & 5 & 8 \\ 0 & 0 & 0 & 0 & 1 & 10 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 11 & 0 & -138 \\ 0 & 0 & 1 & 2 & 0 & -42 \\ 0 & 0 & 0 & 0 & 1 & 10 \end{array}\right]$$

So we get

$$\begin{cases} x_1 + 2x_2 + 11x_4 = -138 \\ x_3 + 2x_4 = -42 \\ x_5 = 10 \end{cases}$$

If we let $x_2 = s, s_4 = t$, then the solutions are

$$\begin{cases} x_1 = 138 - 2s - 11t \\ x_2 = s \\ x_3 = -42 - 2t \\ x_4 = t \\ x_5 = 10 \end{cases}$$

where s, t can be any real numbers.